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SEVERAL GENERATING FUNCTIONS USING GENERALIZED LUCAS SEQUENCES

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Abstract: In this paper I have obtained the generating functions up to third order of generalized sequences defined by Goksal Bilgici. Also I have presented several generating functions of several sequences as particular cases.

Keywords and Phrases: Generating functions, generalized sequences.

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1. Introduction

Many authors [1, 3] generalized sequences differently. In [2] Goksal Bilgici defined generalized sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$. We can write l_n after some modification as follows:

$$l_n = 2al_{n-1} - (a^2 - b)l_{n-2} \qquad n \geq 2 \tag{1.1}$$

where $l_0 = 2$, $l_1 = 2a$.

Clearly, for $(a, b) = \left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{1}{2}, \frac{9}{4}\right), (1, 2)$ the sequence $\{l_n\}_{n=0}^{\infty}$ reduces the Classical Lucas, Jacobsthal-Lucas and Pell-Lucas sequences, respectively. In this note I have obtained the generating functions up to third order of generalized sequence and hence find

1. Generating functions up to third order of Lucas sequence.
2. Generating functions up to third order of Jacobsthal-Lucas sequence.

3. Generating functions up to third order of Pell-Lucas sequence.
The $\{l_n\}$ can also be expressed by the closed form formula.

$$l_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

where α and β are the roots of equation $x^2 - 2ax + (a^2 - b) = 0$.
So that

$$\alpha = a + \sqrt{b} \ \text{ and } \ \ \beta = a - \sqrt{b} \tag{1.3}$$

This gives

$$\alpha + \beta = 2a, \ \ \alpha\beta = a^2 - b, \ \ \alpha - \beta = 2\sqrt{b} \tag{1.4}$$

2. Generating Functions of $\{l_n\}$

Let us solve second order linear recurrence by method of generating function. Let sequence of integer $\{l_n\}$ defined as follows:

$$l_{n+2} - 2al_{n+1} + (a^2 - b)l_n = 0 \ \ n \geq 0, \tag{2.1}$$

where $l_0 = 2$, and $l_1 = 2a$.

Theorem 2.1. Generating function of sequence of integer $\{l_n\}$ is given by

$$\sum_{n=0}^{\infty} l_n x^n = \frac{A_1}{B_1}, \tag{2.2}$$

where $A_1 = 2(1 - ax)$ and $B_1 = 1 - 2ax + (a^2 - b)x^2$.
Proof. Multiplying x^n on both the sides of (2.1) and taking sum from 0 to ∞.

$$\sum_{n=0}^{\infty} l_{n+2}x^n - 2a\sum_{n=0}^{\infty} l_{n+1}x^n + (a^2 - b)\sum_{n=0}^{\infty} l_n x^n = 0$$

$$\frac{1}{x^2}\left[\sum_{n=0}^{\infty} l_n x^n - l_0 - l_1 x\right] - \frac{2a}{x}\left[\sum_{n=0}^{\infty} l_n x^n - l_0\right] + (a^2 - b)\sum_{n=0}^{\infty} l_n x^n = 0$$

$$\sum_{n=0}^{\infty} l_n x^n = \frac{g(x)}{[1 - 2ax + (a^2 - b)x^2]}, \tag{2.3}$$

where $g(x) = l_0 + (l_1 - 2al_0)x$.

Now since $[1-2ax+(a^2-b)x^2]\sum_{n=0}^{\infty} l_n x^n = g(x)$ solving and neglecting terms contains second and higher power of x. Putting $g(x)$ or alternatively putting initial values in (2.3)

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2(1-ax)}{1-2ax+(a^2-b)x^2} \tag{2.4}$$

Now we proceed to find some more generating functions of $\{l_n\}$.

Let $F(x) = \sum\limits_{n=0}^{\infty} l_n x^n = \dfrac{A_1}{B_1}$ where $A_1 = 2(1-ax)$ and $B_1 = 1-2ax+(a^2-b)x^2$. Then

$$\sum_{n=0}^{\infty} l_{n+1} x^n = \frac{F(x)-l_0}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1} x^n = \frac{1}{x}\left[\frac{A_1}{B_1} - 2\right] \quad \text{Since} \quad l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1} x^n = \frac{P_1}{B_1} \quad \text{where} \;\; P_1 = 2a-2(a^2-b)x \quad \text{and} \;\; B_1 = 1-2ax+(a^2-b)x^2. \tag{2.5}$$

$$\text{Again} \quad \sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1} x^n - l_1\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\frac{P_1}{B_1} - l_1\right]$$

$$\sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\frac{P_1}{B_1} - 2a\right] \quad \text{Since } l_1 = 2a$$

$$\sum_{n=0}^{\infty} l_{n+2} x^n = \frac{P_2}{B_1} \quad \text{where} \quad P_2 = 2(a^2+b) - 2a(a^2-b)x. \tag{2.6}$$

$$\text{So in general} \quad \sum_{n=0}^{\infty} l_{n+k} x^n = \frac{P_k}{B_1} \quad \text{where} \quad P_k = l_k - (a^2-b)l_{k-1}x. \tag{2.7}$$

Particular Cases. Now on setting value of a and b in (2.4) to (2.6)

Generating Function of Lucas Sequence On setting $a=\frac{1}{2},\ b=\frac{5}{4}$	**Generating Function of Jacobsthal - Lucas Sequence On setting** $a=\frac{1}{2},\ b=\frac{9}{4}$	**Generating Function of Pell-Lucas Sequence On setting** $a=1,\ b=2$
$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_n x^n = \frac{2-x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_n x^n = \frac{2(1-x)}{(1-2x-x^2)}$
$\sum_{n=0}^{\infty} L_{n+1} x^n = \frac{1+2x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_{n+1} x^n = \frac{1+4x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_{n+1} x^n = \frac{2(1+x)}{(1-2x-x^2)}$
$\sum_{n=0}^{\infty} L_{n+2} x^n = \frac{3+x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_{n+2} x^n = \frac{5+2x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_{n+2} x^n = \frac{6+2x}{(1-2x-x^2)}$

3. Generating Functions of $\{l_n^2\}$

In this section, again using same method we will find generating functions of $\{l_n^2\}$.

Theorem 3.1. *Generating functions of sequence of integer $\{l_n^2\}$ is given by*

$$\sum_{n=0}^{\infty} l_n^2 x^n = \frac{A_2}{B_2}, \tag{3.1}$$

where $A_2 = 4-4(2a^2+b)x+4a^2(a^2-b)x^2$ and $B_2 = 1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3$.

Proof. To find p^{th} order generating function for $\{l_n\}$ we have to expand $\{l_n^p\}$ by the Binomial theorem for which we will use (1.2). This gives $\{l_n^p\}$ as a linear combination of α^{np}, $\alpha^{n(p-1)}\beta^n, ..., \alpha^n\beta^{n(p-1)}, \beta^{np}$. So this generating function has denominator as $(1-\alpha^p x)$ $(1-\alpha^{p-1}\beta x)...(1-\alpha\beta^{p-1}x)$ $(1-\beta^p x)$. Hence to find second order generating function for $\{l_n\}$ we have to expand $\{l_n^2\}$ by the Binomial theorem for which we will use (1.2). So that we can express as linear combination of $(\alpha-\beta)^2(1-\alpha^2x)(1-\beta^2x)(1-\alpha\beta x)$ and using (1.4) we get denominator of generating functions for $\{l_n\}$ as $B_2 = 1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3$.
Consider

$$\sum_{n=0}^{\infty} l_n^2 x^n = \frac{g(x)}{1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3} \tag{3.2}$$

$$g(x) = \left[1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3\right]\sum_{n=0}^{\infty} l_n^2 x^n$$

Considering power of x up to two and neglecting higher powers

$$g(x) = 4 - 4(2a^2 + b)x + 4a^2(a^2 - b)x^2$$

Substituting value of $g(x)$ in (3.2) we get required result. Now we proceed to find some more generating functions of $\{l_n^2\}$.

Let $F_1(x) = \sum_{n=0}^{\infty} l_n^2 x^n = \frac{A_2}{B_2}$ where $A_2 = 4 - 4(2a^2 + b)x + 4a^2(a^2 - b)x^2$ and $B_2 = 1 - (3a^2 + b)x + (a^2 - b)(3a^2 + b)x^2 - (a^2 - b)^3 x^3$. Then

$$\sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{F_1(x) - l_0^2}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{1}{x}\left[\frac{A_2}{B_2} - 4\right] \quad \text{Since} \quad l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{Q_2}{B_2} \quad \text{where} \quad Q_2 = 4a^2 - 4(2a^4 - a^2 b - b^2)x + 4(a^2 - b)^3 x^2. \tag{3.3}$$

Again

$$\sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1}^2 x^n - l_1^2\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\frac{Q_2}{B_2} - l_1^2\right]$$

$$\sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\frac{Q_2}{B_2} - 4a^2\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{Q_3}{B_2} \tag{3.4}$$

where $Q_3 = 4a^2 - (a^2 - b)(3a^2 + b)x + (a^2 - b)^3 x^2$.

Particular Cases. On setting value of a, b in (3.1), (3.3) and (3.4).

Generating Function of Lucas Sequence On setting $a = \frac{1}{2},\ b = \frac{5}{4}$	**Generating Function of Jacobsthal-Lucas Sequence On setting** $a = \frac{1}{2},\ b = \frac{9}{4}$	**Generating Function of Pell-Lucas Sequence On setting** $a = 1,\ b = 2$
$\sum_{n=0}^{\infty} L_n^2\, x^n = \frac{4 - 7x - x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_n^2\, x^n = \frac{4 - 11x - 2x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_n^2\, x^n = \frac{4 - 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$
$\sum_{n=0}^{\infty} L_{n+1}^2\, x^n = \frac{1 + 7x - 4x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_{n+1}^2\, x^n = \frac{1 + 22x - 32x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_{n+1}^2\, x^n = \frac{4 + 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$
$\sum_{n=0}^{\infty} L_{n+2}^2\, x^n = \frac{9 - 2x - x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_{n+2}^2\, x^n = \frac{25 - 26x - 8x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_{n+2}^2\, x^n = \frac{36 + 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$

4. Generating Functions of $\{l_n^3\}$

In this section, again using same method generating functions of $\{l_n^3\}$ is obtained.

Theorem 4.1. *Generating function of sequence of integer $\{l_n^3\}$ is given by*

$$\sum_{n=0}^{\infty} l_n^3 x^n = \frac{A_3}{B_3}, \tag{4.1}$$

where $A_3 = x + 4a(a^2-b)x^2 + (a^2-b)^3x^3$ and
$B_3 = 1-4a(a^2+b)x+(6a^6+2a^4b-6a^2b^2-2b^3)x^2-(4a^9+8a^3b^3-8a^7b-4ab^4)x^3+(a^{12}+b^6+15a^8b^2+15a^4b^4-20a^6b^3-6a^{10}b-6a^2b^5)x^4$.

Proof. To find third order generating functions for $\{l_n^3\}$ we have to expand $\{l_n^3\}$ by the Binomial theorem for which we will use (1.2). Consider

$$\sum_{n=0}^{\infty} l_n^3 x^n = \frac{g(x)}{1-4a(a^2+b)x+(6a^6+2a^4b-6a^2b^2-2b^3)x^2-(4a^9+8a^3b^3-8a^7b-4ab^4)x^3+(a^{12}+b^6+15a^8b^2+15a^4b^4-20a^6b^3-6a^{10}b-6a^2b^5)x^4} \tag{4.2}$$

$g(x) = [1-4a(a^2+b)x+(6a^6+2a^4b-6a^2b^2-2b^3)x^2-(4a^9+8a^3b^3-8a^7b-4ab^4)x^3+(a^{12}+b^6+15a^8b^2+15a^4b^4-20a^6b^3-6a^{10}b-6a^2b^5)x^4]\sum_{n=0}^{\infty} l_n^3x^n$
Considering power of x up to three and neglecting higher powers

$$g(x) = 8-8a(3a^2+4b)x+8(3a^6-b^3-3a^2b^2+a^4b)x^2-8(a^9-a^3b^3-3a^7b+3a^5b^2)x^3$$

Substituting value of $g(x)$ in (4.2) we get required result. Now we proceed to find some more generating functions of $\{l_n^3\}$.
Let

$$F_2(x) = \sum_{n=0}^{\infty} l_n^3 x^n = \frac{A_3}{B_3}$$

where $A_3 = 8-8a(3a^2+4b)x+8(3a^6-b^3-3a^2b^2+a^4b)x^2-8(a^9-a^3b^3-3a^7b+3a^5b^2)x^3$ and
$B_3 = 1-4a(a^2+b)x+(6a^6+2a^4b-6a^2b^2-2b^3)x^2-(4a^9+8a^3b^3-8a^7b-4ab^4)x^3+(a^{12}+b^{16}+15a^8b^2+15a^4b^4-20a^6b^3-6a^{10}b-6a^2b^5)x^4$
Then

$$\sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{F_2(x)-l_0^3}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{1}{x}\left[\frac{A_3}{B_3}-8\right] \quad \text{Since } \; l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{R_3}{B_3} \tag{4.3}$$

where $R_3 = 8a^3 - 8(3a^6 + a^4 b - 3a^2 b^2 - b^3)x + 8(3a^9 + 9a^3 b^3 - 5a^7 b - 3a^5 b^2 - 4ab^4)x^2 - 8(a^{12} + b^6 + 15a^8 b^2 + 15a^4 b^4 - 20a^6 b^3 - 6a^{10} b - 6a^2 b^5)x^3$.
Again

$$\sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1}^3 x^n - l_1^3\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\frac{R_3}{B_3} - l_1^3\right]$$

$$\sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\frac{R_3}{B_3} - 8a^3\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{R_4}{B_3} \quad \text{Since} \quad l_1 = 2a \tag{4.4}$$

where $R_4 = 8(a^6 + b^3 + 3a^2 b^2 + 3a^4 b) - 8(3a^9 - 11a^3 b^3 + 7a^7 b - 3a^5 b^2 + 4ab^4)x + 8(3a^{12} - b^6 - 15a^8 b^2 - 19a^4 b^4 + 28a^6 b^3 - 2a^{10} b + 6a^2 b^5)x^2 - 8(a^{15} + a^3 b^6 + 15a^{11} b^2 + 15a^7 b^4 - 20a^9 b^3 - 6a^{13} b - 6a^5 b^5)x^3$.

Particular Cases: Now setting value of a, b in (4.1), (4.3) and (4.4).

Generating Function of Lucas Sequence On setting $a = 1/2,\ b = 5/4$	**Generating Function of Jacobsthal-Lucas Sequence** On setting $a = 1/2,\ b = 9/4$	**Generating Function of Pell-Lucas Sequence** On setting $a=1,\ b=2$
$\sum_{n=0}^{\infty} L_n^3 x^n = \frac{x - 2x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4}$	$\sum_{n=0}^{\infty} j_n^3 x^n = \frac{8 - 39x - 120x^2 + 8x^3}{1 - 5x - 30x^2 + 40x^3 + 64x^4}$	$\sum_{n=0}^{\infty} Q_n^3 x^n = \frac{8 - 88x - 120x^2 + 8x^3}{1 - 12x - 30x^2 + 12x^3 + x^4}$
$\sum_{n=0}^{\infty} L_{n+1}^3 x^n = \frac{1 - 2x - x^2}{1 - 3x - 6x^2 + 3x^3 + x^4}$	$\sum_{n=0}^{\infty} j_{n+1}^3 x^n = \frac{1 - 120x - 312x^2 - 512x^3}{1 - 5x - 30x^2 + 40x^3 + 64x^4}$	$\sum_{n=0}^{\infty} Q_{n+1}^3 x^n = \frac{8 + 120x - 88x^2 - 8x^3}{1 - 12x - 30x^2 + 12x^3 + x^4}$
$\sum_{n=0}^{\infty} L_{n+2}^3 x^n = \frac{1 + 5x - 3x^2 - x^3}{1 - 3x - 6x^2 + 3x^3 + x^4}$	$\sum_{n=0}^{\infty} j_{n+2}^3 x^n = \frac{125 - 282x - 552x^2 - 64x^3}{1 - 5x - 30x^2 + 40x^3 + 64x^4}$	$\sum_{n=0}^{\infty} Q_{n+2}^3 x^n = \frac{216 + 152x - 104x^2 - 8x^3}{1 - 12x - 30x^2 + 12x^3 + x^4}$

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